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Sequences with group products from finite regular semigroups

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Abstract

We find the smallest integer $g_r(n)$ such that for each finite regular semigroup S of order n, every sequence of length $g_r(n)$ of elements from S contains a consecutive subsequence whose product is a group element.

Keywords: Regular semigroup; Sequence; Group element

0. Introduction

In 1996, Loyola [5] found the smallest integer $\alpha(n)$ such that for every finite regular semigroup S of order n, every sequence of length $\alpha(n)$ of elements of S contains a consecutive subsequence whose product is an α-element where $\alpha =$ "idempotent", "core" and "group and core". For arbitrary semigroups of order n, she also found $\alpha(n)$, where $\alpha =$ "regular", "group", "core", "regular and core" and "group and core".

1. Summary

We prove that for every finite regular semigroup S of order n, every sequence of length $g_r(n)$ where

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$$g_{\mathrm{r}}(n)=\begin{cases} 1 & \text{if } n=4,\\ 2^k & \text{if } n=5k,\quad k\geqslant 1,\\ 2^k & \text{if } n=5k+1,\ 5k+2,\quad k\geqslant 0,\\ 2^k & \text{if } n=5k+3,\quad 0\leqslant k\leqslant 2,\\ \left(\dfrac{9}{8}\right)2^k & \text{if } n=5k+3,\quad k\geqslant 3,\\ \left(\dfrac{3}{2}\right)2^k & \text{if } n=5k+4,\quad k\geqslant 1 \end{cases}$$

contains a consecutive subsequence whose product is a group element and that $g_{\mathrm{r}}(n)$ is the smallest such integer.

2. Preliminaries

Please see [2] for definition of other terms.

Let S be a semigroup. An element a of S is *regular* if for some $x\in S$, $axa=a$. An element e of S is *idempotent* if $e^2=e$. An element g of S is a *group element* if S has a subgroup which contains g. Each singleton with idempotent element is a subgroup of S so that each idempotent is a group element.

For each finite regular semigroup S, we shall denote by $g_{\mathrm{r}}(S)$ the smallest positive integer such that every sequence of length $g_{\mathrm{r}}(S)$ of elements of S contains a consecutive subsequence whose product is a group element. We shall denote by $g_{\mathrm{r}}(n)$ the least upper bound of $\{g_{\mathrm{r}}(S):|S|=n\}$. If S is completely 0-simple, we shall write $g_{\mathrm{c}}(S)$ instead of $g_{\mathrm{r}}(S)$ and also denote by $g_{\mathrm{c}}(n)$ the least upper bound of the set $\{g_{\mathrm{c}}(S):|S|=n\}$.

For any semigroup S, we shall denote by $E(S)$ the set of all idempotent elements of S while we shall denote by S^* the partial groupoid obtained from S by removing its (possibly existing) zero.

Definition 2.1 (Auinger [1]). Let X be a partially ordered set. To each $\alpha\in X$, associate a 0-simple semigroup I_α so that $I_\alpha^*\cap I_\beta^*=\emptyset$ whenever $\alpha\neq\beta$. Suppose that for each pair $\alpha\geqslant\beta$ there is a partial homomorphism $\varphi_{\alpha,\beta}:I_\alpha^*\to I_\beta^*$ subject to the following conditions:

(1) $\varphi_{\alpha,\alpha}$ is the identity mapping on I_α^*,
(2) $\varphi_{\alpha,\beta}\varphi_{\beta,\lambda}=\varphi_{\alpha,\lambda}$ whenever $\alpha\geqslant\beta\geqslant\lambda$.
(3) for any $x\in I_\alpha^*, y\in I_\beta^*$, the set $D(x,y)=\{\gamma\leqslant\alpha,\beta|(x\varphi_{\alpha,\gamma})(y\varphi_{\beta,\gamma})\neq 0 \text{ in } I_\gamma\}$ has a greatest element, to be denoted by $\delta(x,y)$.

Let $S=\bigcup_{\alpha\in X} I_\alpha^*$ and define a binary operation on S by $xy=(x\varphi_{\alpha,\delta(x,y)})(y\varphi_{\beta,\delta(x,y)})$, where $x\in I_\alpha^*$, $y\in I_\beta^*$ and the product is computed in $I_{\delta(x,y)}^*$. The resulting groupoid will be denoted by $(X,I_\alpha,\varphi_{\alpha,\beta})$.

Result 2.2 (Auinger [1, Proposition 1]). *For the groupoid* $(X, I_\alpha, \varphi_{\alpha,\beta})$ *given in Definition* 2.1, *the following hold*:

(1) *S is a semigroup.*
(2) *Green's relation $\mathscr{J}$ is given by the partition* $\{I_\alpha^* \mid \alpha \in X\}$.
(3) $S/\mathscr{J} \cong X$ *(as partially ordered sets).*

Definition 2.2 (Hall [3]). A semigroup is *strict* if $\forall e, f \in E(S)$ and $J_e[J_f]$ is the $\mathscr{J}$-class containing $e[f]$,

$$f \leqslant e \Rightarrow (\forall e_1 = e_1^2 \in J_e)(\exists f_1 = f_1^2 \in J_f) f_1 \leqslant e_1.$$

Result 2.3 (Auinger [1, Theorem 1.2]). *A semigroup S is strict regular if and only if* $S \cong (X, I_\alpha, \varphi_{\alpha,\beta})$ *for completely* 0*-simple semigroups* I_α.

Result 2.4. *For any strict regular semigroup S, a^2 is in a subgroup, for each a in S.*

Result 2.5 (Hall [3, Theorem 4.1]). *For each* $n \geqslant 2$, $g_c(n) = d(n)$ *where $d(n)$ is the largest divisor of $n-1$ less than or equal to* $\sqrt{n-1}$.

Result 2.6 (Hall [3, Lemma 4.6]). *For any semigroup S and any ideal of S,*

$$g(S) \leqslant g(I)g(S/I) \leqslant g(|I|)g(|S/I|)I.$$

Definition 2.3 (Hall and Sapir [4]). Let U and V be semigroups. The *tower* $T = U \circ V$ of U and V is the semigroup obtained by taking the disjoint union of U and V, with each of U and V *as a subsemigroup of T* and $\forall u \in U, v \in V, uv = vu = v$.

We note that the tower of regular semigroups is also regular.

Result 2.7 (Hall and Sapir [4, Lemma 4.2]). *For any semigroups U, V, if the sequences*

$$u = \langle u_1, u_2, \ldots, u_{p-1} \rangle, \quad v = \langle v_1, v_2, \ldots, v_{q-1} \rangle$$

from U, V, respectively, have no consecutive subsequences with idempotent products, then in $U \circ V$, the sequence

$$\langle u, v_1, u, v_2, \ldots, u, v_{q-1}, u \rangle$$

of length $pq - 1$ has no consecutive subsequence with idempotent product.

3. Finding $g_r(n)$

Lemma 3.1.

$$(p+1)^2 \geqslant \begin{cases} 5p+4 & \text{for all } p \geqslant 4, \\ 5p+1 & \text{for all } p \geqslant 3. \end{cases}$$

Proof.

$$
\begin{aligned}
(p+1)^2 &= (p+1)(p+1) \\
&\geqslant 5(p+1) \quad (\text{if } p\geqslant 4) \\
&= 5p+5>5p+4.
\end{aligned}
$$

Since $(3+1)^2=16=5(3)+1$, we also have the second inequality. □

Lemma 3.2. *For each* $n\neq 4,9$, *a bound on* $d(n)$ *is given by*

$$
d(n)\leqslant \begin{cases} p & \textit{if } n=5p, \quad p\geqslant 1, \\ p+1 & \textit{if } n=5p+1, 5p+2, 5p+3, \quad p\geqslant 0, \\ p+1 & \textit{if } n=5p+4, \quad p\geqslant 2. \end{cases}
$$

Proof. Now $d(n)\leqslant\sqrt{n}$, so by Lemma 3.1,

$$
d(n)\leqslant(p+1) \text{ if } n= \begin{cases} 5p+1, \quad p\geqslant 3, \\ 5p+2, \quad 5p+3, 5p+4, \quad p\geqslant 4. \end{cases}
$$

Clearly, $\sqrt{5p}\leqslant p$ for all $p\geqslant 5$. For $1\leqslant p\leqslant 4$, it is also clear that $d(5p)=p$ so $d(5p)\leqslant p$ for all $p\geqslant 1$. The values of $d(n)$, p, and $p+1$ for the remaining cases are shown in Table 1, where we see that $d(n)\leqslant p+1$. □

Definition 3.4. For $n\geqslant 1$, we define $\gamma(n)$ by

$$
\gamma(n)= \begin{cases} 1 & \text{if } n=4, \\ 2^k & \text{if } n=5k, \quad k\geqslant 1, \\ 2^k & \text{if } n=5k+1, 5k+2, \quad k\geqslant 0, \\ 2^k & \text{if } n=5k+3, \quad 0\leqslant k\leqslant 2, \\ \left(\dfrac{9}{8}\right)2^k & \text{if } n=5k+3, \quad k\geqslant 3, \\ \left(\dfrac{3}{2}\right)2^k & \text{if } n=5k+4, \quad k\geqslant 1. \end{cases}
$$

We note that γ is non-decreasing: $\gamma(n)\leqslant\gamma(n+1)$ for all n. Our aim now is to show that $g_{\mathrm{r}}(n)=\gamma(n)$.

Lemma 3.5. *For all* $l,m\geqslant 1, \gamma(l)\gamma(m)\leqslant\gamma(l+m)$.

Proof. Because of the symmetry between l and m, for any $a,b\geqslant 1$, once we have dealt with the case $l=a, m=b$, we need not deal with the case $l=b, m=a$. For $l=1,2,$

Table 1

	$n=5p+1$		$n=5p+2$		$n=5p+3$		$n=5p+4$		
p	n	$d(n)$	n	$d(n)$	n	$d(n)$	n	$d(n)$	$p+1$
0	1	1	2	1	3	1	—	—	1
1	6	2	7	1	8	2	—	—	2
2	11	1	12	3	13	1	14	2	3
3	—	—	17	1	18	3	19	1	4

3, 4, we have $\gamma(l)=1$ so $\gamma(l)\gamma(m)=\gamma(m)\leqslant\gamma(l+m)$ since γ is non-decreasing. Thus we can assume $l\geqslant 5$, and, likewise, $m\geqslant 5$. The cases to be considered are as follows:

Case 1: $l=5k, k\geqslant 1$; then $\gamma(l)=2^k$.
Case 2: $l=5k+1, k\geqslant 1$; then $\gamma(l)=2^k$.
Case 3: $l=5k+2, k\geqslant 1$; then $\gamma(l)=2^k$.
Case 4.1: $l=5k+3, k=1,2$; then $\gamma(l)=2^k$.
Case 4.2: $l=5k+3, k\geqslant 3$; then $\gamma(l)=(\frac{9}{8})2^k$.
Case 5: $l=5k+4, k\geqslant 1$; then $\gamma(l)=(\frac{3}{2})2^k$.

In all cases, we only need to use Definition 3.4 to show that $\gamma(l)\gamma(m)\leqslant\gamma(l+m)$. For example, we consider Case 5.

m	$\gamma(m)$	$\gamma(l)\gamma(m)$	$l+m$	$\gamma(l+m)$
$5q+4, q\geqslant 1$	$(\frac{3}{2})2^q$	$(\frac{9}{4})2^{k+q}$	$5(k+q+1)+3$	$(\frac{9}{8})2^{k+q+1}$

From Cases 1–5, we have $\gamma(l)\gamma(m)\leqslant\gamma(l+m)$ for all $l,m\geqslant 1$. □

Lemma 3.6.

(a) *For* $p\geqslant 0, p+1\leqslant 2^p$.
(b) *For* $p\geqslant 1, p\leqslant(\frac{3}{4})2^p$.

Theorem 3.7. *If* S *is a completely* 0*-simple semigroup and* $|S|\neq 5$, *then*

$$g_c(S)\leqslant\gamma(|S|-1).$$

Proof. If $2\leqslant|S|\leqslant 4$, then $g_c(S)=1$ by Result 2.5. Also, $1\leqslant|S|-1\leqslant 3$, whence $\gamma(|S|-1)=1$ by Definition 3.4. Hence, $g_c(S)=\gamma(|S|-1)$ and we can assume that $|S|>5$. The cases to be considered are as follows:

Case 1: $|S|=5k, k\geqslant 2$.
Case 2: $|S|=5k+1, k\geqslant 1$.
Case 3: $|S|=5k+2, k\geqslant 1$.

Case 4: $|S| = 5k+3, k \geqslant 1$.
Case 5: $|S| = 5k+4, k \geqslant 1$.

In all cases, we get the desired result using Definition 3.4 and Lemmas 3.2 and 3.6. For example, we consider Case 5. Suppose $|S| = 5k+4, k \geqslant 1$

If $k=1$, then $|S|=9$, $g_c(S) \leqslant d(8) = 2 = \gamma(8) = \gamma(|S|-1)$.

If $k=2$ then $|S|=14$, $g_c(S) \leqslant d(13) = 1 < 4 = \gamma(13) = \gamma(|S|-1)$.

Suppose $k \geqslant 3$. By Definition 3.4, $\gamma(|S|-1) = \gamma(5k+3) = \left(\frac{9}{8}\right)2^k$. By Result 2.5, $g_c(S) \leqslant d(5k+3) \leqslant k+1$ by Lemma 3.2. By Lemma 3.6(b), $k+1 \leqslant 2^k$ whence

$$g_c(S) \leqslant 2^k < (\tfrac{9}{8})2^k = \gamma(|S|-1). \qquad \square$$

Lemma 3.8. *Let $S = S^\circ$ be a finite regular semigroup with exactly two non-zero $\mathscr{J}$-classes B and C, say, where $B \leqslant C$ in $S/\mathscr{J}$, and $|B| = 4$. Then either C is a union of groups or B is a union of groups or S is a strict regular semigroup.*

Proof. Suppose that neither B nor C is a union of groups. Let $e, f \in E(C)$ such that $ef \in B$. Since $e \mathscr{J} f$, $eSe \cong fSf$ by Theorem 2.20 [2]. Suppose S is not strict. Then there exist distinct $g, h \in E(B)$ such that $e \geqslant g, h$. Since B is not a union of groups, there is an $\mathscr{H}$-class in B with no idempotent element. Hence, there is exactly one pair of idempotents in B which are not $\mathscr{R}$-related and not $\mathscr{L}$-related. Also, there is at most one pair of idempotents in B which are $\mathscr{R}$-related and at most one pair of idempotents in B which are $\mathscr{L}$-related. Since $eSe \cong fSf$, we have $f \geqslant g, h$ also. Let $(ef)^n$ be the idempotent that is a power of ef. Then $(ef)^n \in B \cup \{0\}$ and clearly, $(ef)^n \geqslant g, h, 0$, since, for example, $fg = g, eg = g$, and so $(ef)^n g = g$. But there is no idempotent element in $B \cup \{0\}$ that is above three idempotents in $B \cup \{0\}$, a contradiction. Hence, S is strict. $\square$

Corollary 3.9. *Let $S = S^\circ$ be a finite regular semigroup with three non-zero $\mathscr{J}$-classes A, B and C, say, where $A \leqslant B \leqslant C$ in $S/\mathscr{J}$, $\|A| = |B| = 4$. Then either A is a union of groups or B is a union of groups or C is a union of groups or S is a strict regular semigroup.*

Lemma 3.10. *Let $S = S^\circ = [J_9/I_5]$ be a regular semigroup where J_9 is a $\mathscr{D}$-class of S with nine elements, I_5 is a 0-minimal ideal with five elements and $I_4 = I_5 \backslash \{0\} \leqslant J_9$. Then $g_r(S) \leqslant 4 = \gamma(|S|-1)$.*

Proof. Now $g_r(S/I_5) \leqslant d(9) = 3$ and $g_r(I_5) \leqslant d(4) = 2$. If I_4 or J_9 is a union of groups, then $g_r(I_5) = 1$ or $g_r(S/I_5) = 1$, whence $g_r(S) \leqslant g_r(I_5) g_r(S/I_5) \leqslant 3$. Suppose that neither J_9 nor I_4 is a union of groups. By Lemma 3.8, S is strict, whence there exists a partial homomorphism $\varphi : J_9 \to I_4$ as in Result 2.3. Take any sequence $s = a_1, a_2, a_3, a_4$ of non-group elements from S of length 4. If the product of some consecutive subsequence s' of s length 3 is in J_9, then every element from this subsequence is in J_9. Since J_9^o is completely 0-simple, $g_c(J_9^o) \leqslant 3$, whence the product of some consecutive subsequence of s' (hence of s) is a group element. So we suppose that $a_1a_2a_3$ and $a_2a_3a_4$ are both in I_5 and $a_1a_2a_3 \neq 0$ and $a_2a_3a_4 \neq 0$.

Case 1: $a_1a_2 \in I_4, a_3a_4 \in I_4$. Since $g_r(I_5) \leqslant 2$, the product of some consecutive subsequence of a_1a_2, a_3a_4, and hence of s, is in a subgroup of I_5.

Case 2: $a_1a_2 \in I_4, a_3a_4 \in J_9$.

Case 2.1: $a_1 \in I_4$. Since $a_2a_3a_4 \in I_5$, the product of some consecutive subsequence of $a_1, a_2a_3a_4$, and hence of s, is a group element.

Case 2.2.1: $a_1 \in J_9, a_2 \in I_4$. From Result 2.4, we may assume that $a_3 \neq a_4$. Since $a_3a_4 \in J_9, a_3a_4 \neq 0$ in J_9^o. Since J_9^o is completely 0-simple, it follows from the proof of Result 2.5 that $(a_3, a_4) \notin \mathscr{R}$ and $(a_3, a_4) \notin \mathscr{L}$. Without loss of generality, we may let the elements of J_9 and I_4 be labeled as shown below. Since $a_3 \mathscr{R} a_3a_4 \mathscr{L} a_4$, we have that j is idempotent.

i	a_3	a_3a_4
l	j	a_4
m	n	k

J_9

g	c
d	h

I_4

Without loss of generality, we may let $a_2 = c$. Since a_2, a_3, a_4, a_3a_4 are non-group elements, none of them is idempotent. Since each $\mathscr{L}$-class and each $\mathscr{R}$-class of S contains an idempotent, it follows that g, h, i and k are all idempotent.

If $a_1\varphi = c$, then $a_1a_2 = (a_1\varphi)a_2 = c^2 = 0$. If $a_1\varphi = d$, then $a_1a_2 = (a_1\varphi)a_2 = dc = h \in E(I_4)$. If $a_1\varphi = h$, then $a_1a_2 = (a_1\varphi)a_2 = hc = 0$. Thus, we can assume that $a_1\varphi = g$.

If $a_3\varphi = c$, then $a_2a_3 = a_2(a_3\varphi) = c^2 = 0$.

If $a_3\varphi = d$, then $a_2a_3 = a_2(a_3\varphi) = cd = g \in E(I_4)$.

If $a_3\varphi = g$, then $a_2a_3 = a_2(a_3\varphi) = cg = 0$.

Thus, we can assume that $a_3\varphi = h$. Since $a_1\varphi = g$, it follows that $a_1 \neq a_3$. Now $a_3a_4 \in J_9$. If $a_4\varphi = g$ then $(a_3a_4)\varphi = (a_3\varphi)(a_4\varphi) = hg = 0$. Hence, $J_9\varphi = \{0\}$, a contradiction to $J_9\varphi \subseteq I_4$. Thus, $a_4\varphi \neq g$ and so $a_1 \neq a_4$. Since $hc = 0$, we likewise have $a_4\varphi \neq c$.

If $a_1 = l$ then $a_3a_1 = i \in J_9$ and $i\varphi = (a_3a_1)\varphi = (a_3\varphi)(a_1\varphi) = hg = 0$, a contradiction. Thus, $a_1 \neq l$.

If $a_1 = m$, then $a_4a_1 = l \in J_9$. If $a_4\varphi = h$, then $l\varphi = (a_4a_1)\varphi = (a_4\varphi)(a_1\varphi) = hg = 0$, whence $a_4\varphi \neq h$. Then $a_4\varphi = d$, which implies that $(a_3a_4)\varphi = (a_3\varphi)(a_4\varphi) = hd = d$. Then, $a_1a_2a_3a_4 = (a_1\varphi)a_2((a_3a_4)\varphi) = gcd = cd = g \in E(I_4)$.

If $a_1 = n$, than $a_4a_1 = j \in J_9$. If $a_4\varphi = h$, $j\varphi = (a_4a_1)\varphi = (a_4\varphi)(a_1\varphi) = hg = 0$, whence $a_4\varphi \neq h$. Then $a_4\varphi = d$ which implies that $(a_3a_4)\varphi = (a_3\varphi)(a_4\varphi) = hd = d$. Then $a_1a_2a_3a_4 = (a_1\varphi)a_2((a_3a_4)\varphi) = gcd = cd = g \in E(I_4)$.

If $a_1 = a_3a_4$, then $a_2a_3a_4 = a_2((a_3a_4)\varphi) = cg = 0$.

Case 2.2.2: $a_1, a_2 \in J_9$. We label the elements of J_9 and I_4 as we did in Case 2.2.1. Since $a_2a_3a_4 \in I_5$, it is clear that $a_2 \neq l, a_2 \neq m$.

Case 2.2.2.1: $a_2 = a_3$. Then $a_2a_3 = (a_3)^2$ is in a subgroup by Result 2.4.

Case 2.2.2.2: $a_2 = a_3a_4$. Then $a_2a_3a_4 = (a_2)^2$ is in a subgroup by Result 2.4.

Case 2.2.2.3: $a_2 = a_4$.

Case 2.2.2.3.1: $a_3\varphi \notin E(I_4)$. Without loss of generality, we may let $a_3\varphi = c$: then $g, h \in E(I_4)$. Since $a_3a_4 \in J_9$, $a_4\varphi \neq c$ (since $c^2 = 0$), and $a_4\varphi \neq g$ (since $cg = 0$).

If $a_4\varphi = d$, then $a_2a_3 = a_4a_3 = (a_4\varphi)(a_3\varphi) = dc = h \in E(I_4)$. If $a_4\varphi = h$, then $a_2a_3 = a_4a_3 = (a_4\varphi)(a_3\varphi) = hc = 0$.

Case 2.2.2.3.2: $a_3\varphi \in E(I_4)$. Without loss of generality, we may let $a_3\varphi = g$.

Suppose $a_4\varphi = c$. If $c \notin E(I_4)$, then $a_2a_3a_4 = a_4a_3a_4 = (a_4\varphi)(a_3\varphi)(a_4\varphi) = cgc = 0$. Otherwise, $a_2a_3a_4 = cgc = c \in E(I_4)$.

If $a_4\varphi = g$, then $a_2a_3a_4 = a_4a_3a_4 = (a_4\varphi)(a_3\varphi)(a_4\varphi) = ggg = g \in E(I_4)$.

Suppose $a_4\varphi = d$. Then $(a_3a_4)\varphi = (a_3\varphi)(a_4\varphi) = gd$. If $d \notin E(I_4)$, then $(a_3a_4)\varphi = gd = 0$, a contradiction to $J_9\varphi \subseteq I_4$. Hence $d \in E(I_4)$ so $a_2a_3a_4 = a_4a_3a_4 = (a_4\varphi)(a_3\varphi)(a_4\varphi) = dgd = dg = d$ is in a group.

Suppose $a_4\varphi = h$. Now $(a_3a_4)\varphi = (a_3\varphi)(a_4\varphi) = gh$. If $d \notin E(I_4)$, then $(a_3a_4)\varphi = 0$, a contradiction, so $d \in E(I_4)$. Then $(a_3a_4)\varphi = gh = c$. Since $a_2 = a_4$, $a_2a_3a_4 = a_4a_3a_4 = (a_4\varphi)((a_3a_4)\varphi) = hc$. If $c \notin E(I_4))$, then $a_2a_3a_4 = hc = 0$. So we suppose that $c \in E(I_4)$. Then $a_2a_3a_4 = hc = h$. Since $g, c, d \in E(I_4)$ and I_4 is not a union of groups, it follows that $h \notin E(I_4)$.

If $a_1\varphi = g$, then $a_1a_2a_3a_4 = (a_1\varphi)a_2a_3a_4 = gh = c \in E(I_4)$.

If $a_1\varphi = c$, then $a_1a_2a_3a_4 = (a_1\varphi)a_2a_3a_4 = ch = 0$.

If $a_1\varphi = h$, then $a_1a_2a_3a_4 = (a_1\varphi)a_2a_3a_4 = hh = 0$.

If $a_1\varphi = d$, then $a_1a_2a_3 = a_1a_4a_3 = (a_1\varphi)(a_4\varphi)(a_3\varphi) = dhg = hg = d \in E(I_4)$.

Case 2.2.2.4: $a_2 = n$.

Case 2.2.2.4.1: $a_3\varphi \notin E(I_4)$. Again, without loss of generality, we let $a_3\varphi = c$: then $g, h \in E(I_4)$. Since $(a_4)^2 \in J_9, a_4\varphi \neq c$ (since $c^2 = 0$), $a_4\varphi \neq g$ (since $cg = 0$).

Suppose $a_4\varphi = d$. Then $(a_3a_4)\varphi = (a_3)\varphi(a_4)\varphi = cd = g$. Now $k \in E(I_9)$ so $a_3 = (a_3a_4)n = (a_3a_4)a_2$ and $a_3\varphi = ((a_3a_4)\varphi)(a_2\varphi)$.

If $a_2\varphi = d$, then $c = a_3\varphi = ((a_3a_4)\varphi)(a_2\varphi) = gd$. If $d \in E(I_4)$ then $c = gd = g$, a contradiction. If $d \notin E(I_4)$, then $c = gd = 0$, also a contradiction. Hence, $a_2\varphi \neq d$.

If $a_2\varphi = g$, then $c = a_3\varphi = ((a_3a_4)\varphi)(a_2\varphi) = gg = g$, a contradiction whence $a_2\varphi \neq g$.

If $a_2\varphi = c$, then $a_2a_3 = (a_2\varphi)(a_3\varphi) = c^2 = 0$.

If $a_2\varphi = h$, then $a_2a_3 = (a_2\varphi)(a_3\varphi) = hc = 0$.

Suppose $a_4\varphi = h$. Since $a_4a_2 \in J_9, a_2\varphi \neq c$ (since $hc = 0$) and $a_2\varphi \neq g$ (since $hg = 0$).

If $a_2\varphi = h$, then $a_2a_3 = (a_2\varphi)(a_3\varphi) = hc = 0$.

If $a_2\varphi = d$, then $a_2a_3 = (a_2\varphi)(a_3\varphi) = dc = h \in E(I_4)$.

Case 2.2.2.4.2: $a_3\varphi \in E(I_4)$. Without loss of generality, we may let $a_3\varphi = g$.

Suppose $a_2\varphi = c$. If $c \in E(I_4)$, then $a_2a_3 = (a_2\varphi)(a_3\varphi) = cg = g \in E(I_4)$. If $c \notin E(I_4)$, then $a_2a_3 = (a_2\varphi)(a_3\varphi) = cg = 0$.

If $a_2\varphi = g$, then $a_2a_3 = (a_2\varphi)(a_3\varphi) = gg = g \in E(I_4)$.

Suppose $a_2\varphi = d$. If $d \in E(I_4)$, then $a_2a_3 = (a_2\varphi)(a_3\varphi) = dg = d$ is in a group, so we may assume that $d \notin E(I_4)$. Since S is regular, it follows that $h \in E(I_4)$. Since $j = a_4a_2 \in J_9$, it follows that $a_4\varphi \neq d$ (since $d^2 = 0$), $a_4\varphi \neq h$ (since $dh = 0$) and $a_4\varphi \neq g$ (since $gd = 0$). Then $a_4\varphi = c$, so $(a_3a_4)\varphi = (a_3\varphi)(a_4\varphi) = gc = c$. Then $a_2a_3a_4 = (a_2\varphi)((a_3a_4)\varphi) = dc = h \in E(I_4)$.

Suppose $a_2\varphi = h$. Then $a_2a_3 = (a_2\varphi)(a_3\varphi) = hg$. If $c \notin E(I_4)$, then $a_2a_3 = hg = 0$. Suppose $c \in E(I_4)$. Then $a_2a_3 = hg = d$. If $d \in E(I_4)$, then a_2a_3 is in a group, so we may assume that $d \notin E(I_4)$. Since S is regular, it follows that $h \in E(I_4)$. Since $(a_3a_4)\varphi \neq 0$ and $gd = gh = 0$, it is clear that $a_4\varphi \neq d, h$.

If $a_4\varphi = g$, then $(a_3a_4)\varphi = (a_3)\varphi(a_4)\varphi = gg = g$. Now $a_3 = (a_3a_4)a_2$, so $a_3\varphi = ((a_3a_4)\varphi)(a_2\varphi) = gh = 0$, a contradiction, whence $a_4\varphi \neq g$.

Then $a_4\varphi = c$ and we have $a_2a_3a_4 = (a_2)\varphi(a_3)\varphi(a_4)\varphi = hgc = dc = h \in E(I_4)$.

Case 3: $a_1a_2 \in J_9, a_3a_4 \in I_4$. This case is dual to Case 2.

Case 4: $a_1a_2, a_3a_4 \in J_9$; then $a_1, a_2, a_3, a_4 \in J_9$. We label the elements of J_9 and I_4 as we did in Case 2.2.1. As before, $i, j, k \in E(S)$. If $a_1a_2 = l$, then $a_1a_2a_3a_4 = a_4 \in J_9$. If $a_1a_2 = m$, then $a_1a_2a_3a_4 = k \in J_9$. Since $a_1a_2a_3a_4 \in I_5$, it follows that $a_1a_2 \neq l, m$.

Case 4.1: $a_1a_2 = a_3$. Then $a_1a_2a_3 = (a_3)^2$ is in a subgroup by Result 2.4.

Case 4.2: $a_1a_2 = a_3a_4$. Then $a_1a_2a_3a_4 = (a_3a_4)^2$ is in a subgroup by Result 2.4.

Case 4.3: $a_1a_2 = a_4$.

Case 4.3.1: $a_3\varphi \notin E(I_5)$. Without loss of generality, we may let $a_3\varphi = c$: then $g, h \in E(I_4)$. Since $a_3a_4 \in J_9$, $a_4\varphi \neq c$ (since $c^2 = 0$), $a_4\varphi \neq g$ (since $cg = 0$).

If $a_4\varphi = d$, then $a_1a_2a_3 = ((a_1a_2)\varphi)(a_3\varphi) = dc = h \in E(I_4)$.

If $a_4\varphi = h$, then $a_1a_2a_3 = ((a_1a_2)\varphi)(a_3\varphi) = hc = 0$.

Case 4.3.2: $a_3\varphi \in E(I_5)$. Without loss of generality, we may let $a_3\varphi = g$.

If $a_4\varphi = c$, then $a_1a_2a_3 = ((a_1a_2)\varphi)(a_3\varphi) = cg$. If $c \in E(I_4)$, then $a_1a_2a_3 = cg = g \in E(I_4)$. Otherwise, $a_1a_2a_3 = cg = 0$.

If $a_4\varphi = g$, then $a_1a_2a_3 = ((a_1a_2)\varphi)(a_3\varphi) = gg = g \in E(I_4)$.

If $a_4\varphi = d$, then since $a_3a_4 \in J_9$ and $(a_3a_4)\varphi = (a_3)\varphi(a_4)\varphi = gd$, it follows that $d \in E(I_4)$. Then $a_1a_2a_3 = ((a_1a_2)\varphi)(a_3\varphi) = dg = d$ is in a group.

Suppose $a_4\varphi = h$. If $d \notin E(I_4)$, then $(a_3a_4)\varphi = (a_3)\varphi(a_4)\varphi = gh = 0$, a contradiction so $d \in E(I_4)$.

If $h \in E(I_4)$, then since I_4 is not a union of groups, it follows that $c \notin E(I_4)$. Then $a_1a_2a_3 = ((a_1a_2)\varphi)(a_3\varphi) = hg = 0$.

Suppose $h \notin E(I_4)$. Since each $\mathscr{L}$-class contains an idempotent, it follows that $c \in E(I_4)$. Now $a_1a_2 = a_4$, so $(a_1a_2)\varphi = a_4\varphi = h$. Then either

(1) $(a_1)\varphi = d$ and $(a_2)\varphi = h$ or
(2) $(a_1)\varphi = h$ and $(a_2)\varphi = c$ or
(3) $(a_1)\varphi = d$ and $(a_2)\varphi = c$.

If $(a_1)\varphi = d$ and $(a_2)\varphi = h$, then $a_1a_2a_3 = (a_1\varphi)(a_2\varphi)(a_3\varphi) = dhg = dd = d \in E(I_4)$.

If $(a_2)\varphi = c$, then $a_2a_3 = (a_2\varphi)(a_3\varphi) = cg = g \in E(I_4)$.

Case 4.4: $a_1a_2 = n$.

Case 4.4.1: $a_3\varphi \notin E(I_5)$. Without loss of generality, we may let $a_3\varphi = c$: then g, $h \in E(I_4)$. Since $a_3a_4 \in J_9$, $a_4\varphi \neq c$ (since $c^2 = 0$), $a_4\varphi \neq g$ (since $cg = 0$). Now $k \in E(J_9)$ so $a_3 = (a_3a_4)n = (a_3a_4)(a_1a_2)$ and $j = a_4(a_1a_2)$. Since $j \in E(J_9)$, $k = na_4 = (a_1a_2)a_4$.

If $a_4\varphi = d$, then $(a_3a_4)\varphi = (a_3)\varphi(a_4)\varphi = cd = g$. Also, $c = (a_3)\varphi = (a_3a_4)\varphi(a_1a_2)\varphi = g((a_1a_2)\varphi)$ so $(a_1a_2)\varphi = c$. Then $a_1a_2a_3 = ((a_1a_2)\varphi)(a_3\varphi) = c^2 = 0$. Suppose $a_4\varphi = h$. Since $a_4(a_1a_2) \in J_9$, $(a_1a_2)\varphi \neq c$ (since $hc = 0$), and $(a_1a_2)\varphi \neq g$ (since $hg = 0$).

If $(a_1a_2)\varphi = h$, then $a_1a_2a_3 = ((a_1a_2)\varphi)(a_3\varphi) = hc = 0$.

If $(a_1a_2)\varphi = d$, then $a_1a_2a_3 = ((a_1a_2)\varphi)(a_3\varphi) = dc = h \in E(I_4)$.

Case 4.4.2: $a_3\varphi \in E(I_5)$. Without loss of generality, we may let $a_3\varphi = g$.

If $(a_1a_2)\varphi = c$, then $a_1a_2a_3 = ((a_1a_2)\varphi)(a_3\varphi) = cg$. If $c \in E(I_4)$, then $a_1a_2a_3 = (a_1a_2\varphi)(a_3\varphi) = cg = g \in E(I_4)$. If $c \notin E(I_4)$ then $a_1a_2a_3 = cg = 0$.

If $(a_1a_2)\varphi = g$, then $a_1a_2a_3 = ((a_1a_2)\varphi)(a_3\varphi) = gg = g \in E(I_4)$.

If $(a_1a_2)\varphi = d$, then $a_1a_2a_3 = ((a_1a_2)\varphi)(a_3\varphi) = dg = d$. If $d \in E(I_4)$, then $a_1a_2a_3 = d$ is in a group, so we assume that $d \notin E(I_4)$. Since each $\mathscr{R}$-class contains an idempotent, it follows that $h \in E(I_4)$.

Now $a_4(a_1a_2) = j \in J_9$. Hence, $a_4\varphi \neq d$ (since $d^2 = 0$) and $a_4\varphi \neq g$ (since $gd = 0$).

Since $(a_1a_2)a_4 = k \in J_9$, we also have that $a_4\varphi \neq h$ (since $dh = 0$).

Thus, $a_4\varphi = c$ and we have $(a_3a_4)\varphi = (a_3)\varphi(a_4)\varphi = gc = c$. Then $a_1a_2a_3a_4 = ((a_1a_2)\varphi)((a_3a_4)\varphi) = dc = h \in E(I_4)$.

Suppose $(a_1a_2)\varphi = h$. Then $a_1a_2a_3 = ((a_1a_2)\varphi)(a_3\varphi) = hg$. If $c \notin E(I_4)$, then $a_1a_2a_3 = hg = 0$. Thus, we may assume that $c \in E(I_4)$. Then $a_1a_2a_3 = hg = d$. If $d \in E(I_4)$, then $a_1a_2a_3$ is in a group so we assume that $d \notin E(I_4)$. Since each $\mathscr{R}$-class contains an idempotent, it follows that $h \in E(I_4)$.

Now $(a_3a_4)\varphi = (a_3)\varphi(a_4)\varphi = g(a_4\varphi)$. Since $gd = gh = 0$, it follows that $a_4\varphi \neq d, h$. Also, $a_4(a_1a_2) = j \in J_9$ so $j\varphi = (a_4\varphi)((a_1a_2)\varphi) = (a_4\varphi)h$. Since $gh = 0$, we have $a_4\varphi \neq g$.

Then $a_4\varphi = c$ so $(a_3a_4)\varphi = (a_3)\varphi(a_4)\varphi = gc = c$. Thus, $a_1a_2a_3a_4 = ((a_1a_2)\varphi)((a_3a_4)\varphi) = hc = c \in E(I_4)$

From Cases 1–4, the product of some consecutive subsequence of $s = a_1, a_2, a_3, a_4$ is in a subgroup, whence $g_r(S) \leqslant 4$, as required. □

Corollary 3.11. *Let*

$$S = S^\circ = \begin{bmatrix} K_9 \\ \overline{J_4} \\ \overline{I_5} \end{bmatrix}$$

be a regular semigroup where J_4 *and* K_9 *are* $\mathscr{D}$*-classes of* S *with* 4 *and* 9 *elements, respectively,* I_5 *is a* 0*-minimal ideal with* 5 *elements,* $I_5\backslash\{0\} \leqslant J_4 \leqslant K_9$. *Then* $g_r(S) \leqslant 8 = \gamma(|S| - 1)$.

Proof. By Lemma 3.10,

$$g_r(S/I_5) = g_r \begin{bmatrix} K_9 \\ \overline{J_4} \\ \bar{0} \end{bmatrix} \leqslant 4.$$

By Result 2.6,

$$g_r(S) = g_r \begin{bmatrix} K_9 \\ \overline{J_4} \\ \overline{I_5} \end{bmatrix} \leqslant g_r(I_5)g_r(S/I_5) \leqslant 2.4 = 8 = \gamma(|S| - 1),$$

by Definition 3.4. □

Table 2

$\lvert J\rvert$	$\lvert S\rvert$	$\lvert S\rvert - 1$	$\overline{g_r(S)}$	$\gamma(\lvert S\rvert - 1)$
$5p, p\geqslant 1$	$5p+5$	$5p+4$	$2p$	$(\frac{3}{2})2^p$
$5p+1, p\geqslant 0$	$5(p+1)+1$	$5(p+1)$	$2(p+1)$	2^{p+1}
$5p+2, p\geqslant 0$	$5(p+1)+2$	$5(p+1)+1$	$2(p+1)$	2^{p+1}
$5p+3, p\geqslant 0$	$5(p+1)+3$	$5(p+1)+2$	$2(p+1)$	2^{p+1}
$5p+4, p\geqslant 2$	$5(p+1)+4$	$5(p+1)+4$	$2(p+1)$	$(\frac{9}{8})2^{p+1}$
9	14	13	4	4

Lemma 3.12. *For any finite regular semigroup $S=S^\circ$ which is the union of a* 0*-minimal ideal I with $|I|=5$, and a $\mathscr{J}$-class J (disjoint from I) such that $I\backslash\{0\}\leqslant J$ (in the partial ordering of $\mathscr{J}$-classes) and $|J|\neq 4$, $g_r(S)\leqslant\gamma(|S|-1)$.*

Proof. Since I is 0-minimal, I is a completely 0-simple subsemigroup of S so $g_r(I)=g_c(I)\leqslant d(4)=2$ by Result 2.5. We also note that S/I is a completely 0-simple semigroup so $g_r(S/I)=g_c(|S/I|)=d(|S/I|-1)=d(|J|)$ also by Result 2.5. By Lemma 3.2,

$$d(|J|)\leqslant\begin{cases} p & \text{if } |J|=5p, \quad p\geqslant 1,\\ p+1 & \text{if } |J|=5p+1, 5p+2, 5p+3, \quad p\geqslant 0,\\ \mathrm{p}+1 & \text{if } |J|=5p+4, \quad p\geqslant 2,\end{cases}$$

whence by Result 2.6, $g_r(S)\leqslant g_r(I)g_r(S/I)\leqslant 2d(|J|)$.

If $|J|=9$, it follows from Lemma 3.10 that $g_r(S)\leqslant 4$ whence

$$g_r(S)\leqslant\begin{cases} 2p & \text{if } |J|=5p, \ p\geqslant 1,\\ 2(p+1) & \text{if } |J|=5p+1, 5p+2, 5p+3, \quad p\geqslant 0,\\ 2(p+1) & \text{if } |J|=5p+4, \quad p\geqslant 2,\\ 4 & \text{if } |J|=9.\end{cases} \tag{$*$}$$

In Table 2, $\overline{g_r(S)}$ denotes an upper bound for $g_r(S)$ which we obtained from $(*)$.

From Lemma 3.6, in each of the six rows of the table, we have $g_r(S)\leqslant\gamma(|S|-1)$. Thus, $g_r(S)\leqslant\gamma(|S|-1)$ as required. □

Lemma 3.13. *Let $S=S^\circ=[J:K/0]$ be a finite regular semigroup with two* 0*-minimal ideals J and K and such that $S=J\cup K$. Then $g_r(S)\leqslant\max\{d(|J|-1), d(|K|-1)\}$.*

Proof. Let $p=\max\{d(|J|-1), d(|K|-1)\}$ and take any sequence $s=a_1, a_2, \ldots, a_p$ of elements from S of length p. Since S is regular, $JK=KJ=J\cap K=\{0\}$ by Exercise 1.9.11 [2]. It follows that if $a_i\in J$ and $a_j\in K$ for some i, j, then the product $a_1a_2\ldots a_p=0$ a group element. Thus, we can assume that either $\{a_1, a_2, \ldots, a_p\}\subset J$

or $\{a_1, a_2, \ldots, a_p\} \subset K$. But J and K are completely 0-simple semigroups, and so

$$g_c(J) \leqslant d(|J|-1) \leqslant p \quad \text{and} \quad g_c(K) \leqslant d(|K|-1) \leqslant p.$$

Thus, the product of some consecutive subsequence of s is a group element, as required. □

Corollary 3.14. *Let $S = S^\circ = [J_4 : K_4/I_5]$ be a finite regular semigroup with (non-zero) $\mathscr{D}$-classes I_4, J_4 and K_4, where $|I_4| = |J_4| = |K_4| = 4$, $I_4 \leqslant J_4$, $I_4 \leqslant K_4$, $J_4 \nleqslant K_4$ and $K_4 \nleqslant J_4$. Then $g_r(S) \leqslant \gamma(|S|-1)$.*

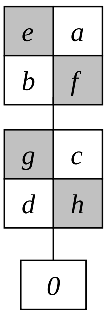
Proof. Since I_5 is completely 0-simple, $g_c(I_5) \leqslant 2$ by Result 2.5. By Lemma 3.13, $g_r(S/I_5) = g_r[J_4 : K_4/0] \leqslant \max\{d(|J_4|), d(|K_4|)\} = \max\{d(4), d(4)\} = 2$. By Result 2.6,

$$g_r(S) \leqslant g_r(I_5) g_r(S/I_5) \leqslant 2 \cdot 2 = 4.$$

By Definition 3.4, $\gamma(12) = 4$, whence $g_r(S) \leqslant 4 = \gamma(12) = \gamma(|S|-1)$. □

Lemma 3.15. *Let $S = S^\circ$ be a strict regular semigroup with two non-zero $\mathscr{J}$-classes A and B such that $A \leqslant B$ and $|A| = |B| = 4$. Let $\varphi : B \to A$ be a partial homomorphism as in Result 2.3. If φ is onto, then $g_r(S) \leqslant 2$.*

Proof. If A or B is a union of groups, we have $g_r(S) \leqslant g_r(I_5) g_r(S/I_5) \leqslant 2$. So we suppose neither A nor B is a union of groups. Let $s = a_1, a_2$ be a sequence of non-group elements from S. By Result 2.4, we may assume that $a_1 \neq a_2$. If a_1 and a_2 both belong to the same $\mathscr{J}$-class, then since a_1 and a_2 are non-group elements, their product $a_1 a_2$ is in a group in that $\mathscr{J}$-class. Hence, we only need to consider the case when $a_1 \in B, a_2 \in A$ (the other case being dual). Without loss of generality, we label the elements of A and B as shown on the right.



Since φ is onto A and $|A| = |B|$, φ is also one to one. Without loss of generality, we let $e\varphi = g, a\varphi = c, b\varphi = d, f\varphi = h$. Since B is not a union of groups, we may assume that a is not in a group. Since each $\mathscr{R}$-class and each $\mathscr{L}$-class contains an idempotent, it follows that $e, f \in E(B)$. Since φ maps idempotents in B to idempotents in A, we have $g, h \in E(A)$.

Case 1: $b \in E(B)$; then $b\varphi = d \in E(A)$. Since A is not a union of groups and $g, d, h \in E(A)$, it follows that $c \notin E(A)$. Then $a_1 = a, a_2 = c$ and $a_1 a_2 = c^2 = 0$.

Case 2: $b \notin E(B)$.

Case 2.1: $c \in E(A)$. Since A is not a union of groups, it follows that $d \notin E(A)$ and that $a_2 = d$. If $a_1 = a$, then $a_1 a_2 = (a\varphi)d = cd = g \in E(A))$. If $a_1 = b$ then $a_1 a_2 = (b\varphi)d = dd = 0$.

Case 2.2: $c \notin E(A)$. If $a_1 = a$ and $a_2 = c$, then $a_1 a_2 = c^2 = 0$. If $a_1 = b$ and $a_2 = c$, then $a_1 a_2 = (b\varphi)c = dc = h \in E(A)$. If $a_1 = b$ and $a_2 = d$ (so $d \notin E(A)$), we have $a_1 a_2 = d^2 = 0$. If $a_1 = a$ and $a_2 = d$, then $a_1 a_2 = (a\varphi)d = cd = g \in E(A)$.

From Cases 1 and 2, we have $g_r(S) \leqslant 2$ as required. $\square$

Lemma 3.16. *Let $S = S^\circ$ be a strict regular semigroup with two non-zero $\mathscr{J}$-classes A and B such that $A \leqslant B$, $|A| = |B| = 4$ and each of A and B has two $\mathscr{R}$-classes and two $\mathscr{L}$-classes. Let $\varphi : B \to A$ be a partial homomorphism as in Result* 2.3. *If φ is not onto, then either* $\operatorname{ran}\varphi = \{i\}$ *for some $i \in E(A)$ or* $\operatorname{ran}\varphi = \{j, k\}$ *for some $j, k \in E(A)$ such that either $j\mathscr{R}k$* or *$j\mathscr{L}k$.*

Proof. We label the elements of A and B as we did in Lemma 3.15. If B is a union of groups, then B is a rectangular band, whence $B\varphi$ is also a rectangular band. Since $B\varphi \neq A$ we have $|B\varphi| < 4$ whence $B\varphi$, being a morphic image of B is either a trivial semigroup, a two-element left-zero semigroup, or a two-element right-zero semigroup, which gives the desired result. So we can assume that B is not a union of groups. Suppose a is not a group element. Then $e, f \in E(B)$. Let $e\varphi = g$. Since φ maps idempotents to idempotents, $g \in E(A)$. Now $ea = a$ so $g(a\varphi) = a\varphi$. Since $gh \neq h, gh \neq d$, either $a\varphi = c$ or $a\varphi = g$.

Case 1: $a\varphi = c$. Since $af = a$, $c(f\varphi) = c$. Since $cg \neq c$, $cd \neq c$, either $f\varphi = c$ or $f\varphi = h$. If $f\varphi = h$, then $b\varphi = d$ and φ is onto. Thus $f\varphi = c$. Since $f \in E(B)$, $c \in E(A)$. Then $\operatorname{ran}\varphi = \{g, d\}$ where $g, c \in E(A)$ and $g\mathscr{R}c$.

Case 2: $a\varphi = g$. Since $be = b$, $(b\varphi)g = b\varphi$. Since $cg \neq c$ and $hg \neq h$, either $b\varphi = g$ or $b\varphi = d$.

If $b\varphi = g$, then since $ba = f, g = gg = (b\varphi)(a\varphi) = f\varphi$ and $\operatorname{ran}\varphi = \{g\}$.

If $b\varphi = d$, then since $ba = f, d = dg = (b\varphi)(a\varphi) = f\varphi$. Since $f \in E(B)$, $d \in E(A)$. Hence, $\operatorname{ran}\varphi = \{g, d\}$ where $g, d \in E(A)$ and $g\mathscr{L}d$. $\square$

Lemma 3.17. *Let $S = S^\circ = [J_4/I_5] = I_5 \cup J_4$ be a finite regular semigroup with a* 0-*minimal ideal I_5 and a $\mathscr{D}$-class J_4 having four elements such that $I_4 \leqslant J_4$. Then $g_r(S) \leqslant 3$.*

Proof. By Result 2.5, $g_r(I_5) \leqslant 2$ and $g_r(S/I_5) \leqslant 2$. If I_4 or J_4 is a union of groups then $g_r(I_5) = 1$ or $g_r(S/I_5) = 1$, whence $g_r(S) \leqslant g_r(I_5) g_r(S/I_5) \leqslant 2$. So we can assume neither I_4 nor J_4 is a union of groups. By Lemma 3.8, S is strict so by Result 2.3, the ideal extension S of I_5 by J_4 is determined by a partial homomorphism $\varphi : J_4 \to I_4$. If φ is onto, it follows from Lemma 3.15 that $g_r(S) \leqslant 2$. Hence, we may assume that φ is not onto.

Take any sequence $s = a_1, a_2, a_3$ of non-group elements from S of length 3. If at least two of these elements are in I_5, then since I_5 is an ideal, $a_1 a_2 a_3 = b_1 b_2$ where

$b_1, b_2 \in I_5$ and b_1 and b_2 are products of consecutive non-overlapping subsequences of s. If s does not have a consecutive subsequence of length one or two whose product is in a subgroup, then neither b_1 nor b_2 is in a subgroup. Since $g_r(I_5) \leqslant 2$, $a_1a_2a_3 = b_1b_2$ is in a group. Hence, we may assume that there is at most one element in s, which is in I_5.

Let a_j, a_{j+1} be a consecutive subsequence of s such that $a_j, a_{j+1} \in J_4$. If $a_j = a_{j+1}$, then $a_ja_{j+1} = (a_j)^2$ is in a group by Result 2.4. If $a_j \neq a_{j+1}$, then since a_j and a_{j+1} are non-group elements, their product a_ja_{j+1} is in a subgroup of J_4. Hence, we may assume that s has no consecutive subsequence a_j, a_{j+1} of length 2 such that $a_j, a_{j+1} \in J_4$. It now remains to consider the case where $a_1, a_3 \in J_4, a_2 \in I_4$. We label the elements of I_4 and J_4 as we did in the proof of Lemma 3.15. Without loss of generality, let $a_2 = c$. Since c is not a group element, it follows that $g, h \in E(I_4)$.

Case 1: $|\mathrm{ran}\,\varphi| = 1$. If $\mathrm{ran}\,\varphi = \{g\}$, then $a_2a_3 = a_2(a_3\varphi) = cg = 0$. If $\mathrm{ran}\,\varphi = \{d\}$, then $a_1a_2 = (a_1\varphi)a_2 = dc = h \in E(I_4)$. If $\mathrm{ran}\,\varphi = h$, then $a_1a_2 = (a_1\varphi)a_2 = hc = 0$.

Case 2: $|\mathrm{ran}\,\varphi| = 2$.

Case 2.1: $\mathrm{ran}\,\varphi = \{g, d\}$; *thus* $d \in E(A)$. If $a_3\varphi = g$, then $a_2a_3 = cg = 0$. Hence we may let $a_3\varphi = d$. Then $a_2a_3 = a_2(a_3\varphi) = cd = g \in E(A)$.

Case 2.2: $\mathrm{ran}\,\varphi = \{d, h\}$; *thus* $d \in E(A)$. If $a_3\varphi = d$, then $a_2a_3 = cd = g \in E(I_4)$. Hence, we may let $a_3\varphi = h$. If $a_1\varphi = h$, then $a_1a_2 = (a_1\varphi)a_2 = hc = 0$. If $a_1\varphi = d$ then $a_1a_2 = (a_1\varphi)a_2 = dc = h \in E(A)$.

From Cases 1 and 2, the product of some consecutive subsequence of $s = a_1, a_2, a_3$ is a group element, whence $g_r(S) \leqslant 3$. □

Lemma 3.18. *Let*

$$S = S^\circ = \begin{bmatrix} K_4 \\ \overline{J_4} \\ \overline{I_5} \end{bmatrix}$$

be a regular semigroup with three non-zero $\mathscr{J}$*-classes* I_4, J_4 *and* K_4 *with* $|I_4| = |J_4| = |K_4| = 4$, $I_4 \leqslant J_4$, $J_4 \leqslant K_4$, $I_5 = I_4 \cup \{0\}$. *Then* $g_r(S) \leqslant 4 = \gamma(|S| - 1)$.

Proof. Now $I_5 \cup J_4$ is an ideal of S. If I_4 or J_4 is a union of groups, then $g_r(I_5 \cup J_4) \leqslant 2$. By Result 2.6,

$$g_r(S) \leqslant g_r(I_5 \cup J_4) g_r(S/(I_5 \cup J_4)) \leqslant 2 \cdot 2 = 4.$$

Hence, we may assume that neither I_4 nor J_4 is a union of groups. If K_4 is a union of groups, then $g_r(S) = g_r(I_5 \cup J_4) \leqslant 3$ by Lemma 3.17. Hence we may also assume that K_4 is not a union of groups. Then each of I_4, J_4 and K_4 has two $\mathscr{R}$-classes and two $\mathscr{L}$-classes. We note that if $a, b \in S$ are such that $a \neq b$, $a \mathscr{J} b$ and a and b are non-group elements, then $(a, b) \notin \mathscr{R}$, $(a, b) \notin \mathscr{L}$, so $ab, ba \in E(S)$.

Since none of I_4, J_4 and K_4 is a union of groups, it follows from Corollary 3.9 that S is strict. Hence there exist partial homomorphisms $\varphi_{K_4,J_4}, \varphi_{K_4,I_4}$, φ_{J_4,I_4} as in Result 2.3. If φ_{J_4,I_4} is onto, then $= g_r(I_5 \cup J_4) \leqslant 2$ by Lemma 3.15, whence $g_r(S) \leqslant$

$g_r(I_5 \cup J_4)g_r(S/(I_5 \cup J_4)) \leqslant 2 \cdot 2 = 4$ by Result 2.6. So we assume that φ_{J_4,I_4} is not onto.

Take a sequence

$$s = a_1, a_2, a_3, a_4$$

of elements from S. If three (or four) of a_1, a_2, a_3, a_4 are in $I_5 \cup J_4$, then

$$s = s_1, s_2, s_3,$$

where s_1, s_2, s_3 are consecutive subsequences of s, and each s_i contains an element of $I_5 \cup J_4$. Let p_i be the product of s_i, $i = 1, 2, 3$. Consider p_1, p_2, p_3, a sequence in $I_5 \cup J_4$ (since $I_5 \cup J_4$ is an ideal of S). Since $g_r(I_5 \cup J_4) \leqslant 3$, we have that a consecutive subsequence of p_1, p_2, p_3, and hence of s, has a group product. Thus we can assume two (or more) of a_1, a_2, a_3, a_4 are in K_4, say a_i, a_j where $i < j$.

If $a_i a_{i+1} \ldots a_j \in K_4$, then since $g_r(S/(I_5 \cup J_4)) \leqslant 2$, we have that some consecutive subsequence of $a_i a_{i+1} \ldots a_j$ has a group product.

If $a_i a_{i+1} \ldots a_j \notin K_4$, we let φ denote $\varphi_{K_4, J_{a_i a_{i+1} \ldots a_j}}$. Then

$$\begin{aligned} a_i a_{i+1} \ldots a_j &= (a_i \varphi) a_{i+1} \ldots (a_j \varphi) \\ &\in (K_4 \varphi) S^1 (K_4 \varphi) \cap J_{a_i a_{i+1} \ldots a_j} \\ &\subseteq (K_4 \varphi) \\ &\subseteq E(S). \qquad \square \end{aligned}$$

Corollary 3.19. *Let*

$$S = S^\circ = \begin{bmatrix} K \\ \overline{J_4} \\ \overline{I_5} \end{bmatrix}$$

be a regular semigroup with three non-zero $\mathscr{J}$-classes I_4, J_4 and K with $|I_4| = |J_4| = 4$, $I_4 \leqslant J_4 \leqslant K$. Then $g_r(S) \leqslant \gamma(|S| - 1)$.

Proof. Let $I = [J_4/I_5]$. Then I is an ideal of S and by Lemma 3.17, $g_r(I) \leqslant 3$. By Lemma 3.2 we have

$$g_r(S/I) \leqslant d(|K|) \leqslant \begin{cases} p & \text{if } |K| = 5p, \quad p \geqslant 1, \\ p+1 & \text{if } |K| = 5p+1, 5p+2, 5p+3, \quad p \geqslant 0, \\ p+1 & \text{if } |K| = 5p+4, \quad p \geqslant 2. \end{cases}$$

Thus, by Result 2.6,

$$g_r(S) \leqslant g_r(I) g_r(S/I) \leqslant 3d(|K|).$$

Table 3

$\|K\|$	$\|S\|$	$\|S\|-1$	$\overline{g_r(S)}$	$\gamma(\|S\|-1)$
5	14	13	3	4
$5p,\ p\geqslant 2$	$5(p+1)+4$	$5(p+1)+3$	$3p$	$(\frac{9}{8})2^{p+1}$
$5p+1,\ p\geqslant 0$	$5(p+2)$	$5(p+1)+4$	$3(p+1)$	$(\frac{3}{2})2^{p+1}$
$5p+2,\ p\geqslant 0$	$5(p+2)+1$	$5(p+2)$	$3(p+1)$	2^{p+2}
$5p+3,\ p\geqslant 0$	$5(p+2)+2$	$5(p+2)+1$	$3(p+1)$	2^{p+2}
$5p+4,\ p\geqslant 2$	$5(p+2)+3$	$5(p+2)+2$	$3(p+1)$	2^{p+2}
4	13	12	4	4
9	18	17	8	8

If $|K|=4$, then $g_r(S)\leqslant 4(=\gamma(|S|-1))$ by Lemma 3.18. If $|K|=9$, then $g_r(S)\leqslant 8$ $(=\gamma(|S|-1))$ by Corollary 3.11. Thus we have

$$g_r\begin{bmatrix} K \\ J_4 \\ I_5 \end{bmatrix} \leqslant \begin{cases} 3p & \text{if } |K|=5p, \quad p\geqslant 1, \\ 3(p+1) & \text{if } |K|=5p+1, 5p+2, 5p+3, \quad p\geqslant 0, \\ 3(p+1) & \text{if } |K|=5p+4, \quad p\geqslant 2, \\ 4 & \text{if } |K|=4, \\ 8 & \text{if } |K|=9. \end{cases} \tag{$**$}$$

In the Table 3, $\overline{g_r(S)}$ denotes the upper bound for $g_r(S)$ given in $(**)$.

For all eight rows, we see that $g_r(S)\leqslant\overline{g_r(S)}\leqslant\gamma(|S|-1)$ using Lemma 3.6 for rows 2–6. □

Lemma 3.20. *In every finite non-trivial regular semigroup S with $|S|\neq 5$ and $\neq 9$, there is an ideal I such that $g_r(I)\leqslant\gamma(|I|-1)$.*

Proof. If S has no zero, then its least ideal (or kernel) I is a union of groups, and $|I|\geqslant 2$; thus $g_r(I)=1\leqslant\gamma(|I|-1)$.

Suppose $S=S^{\circ}$. Let I' be a 0-minimal ideal of S. Then $|I'|\geqslant 2$. If $2\leqslant|I'|\leqslant 4$, then I' is a union of groups so $g_r(I)=1\leqslant\gamma(|I|-1)$. So we can suppose that $|I'|\geqslant 5$.

Case 1: S has exactly one 0-minimal ideal.

Case 1.1: $|I'|\neq 5$. Take $I=I'$. Since I is completely 0-simple, we have $g_c(I)\leqslant\gamma(|I|-1)$ by Theorem 3.7.

Case 1.2: $|I'|=5$. Since $|S|\neq 5$ we have $S\neq I'$ and since S has only one 0-minimal ideal, there is a $\mathscr{J}$-class J in S such that $I'\backslash\{0\}=I_4\leqslant J$ (in the partial ordering of $\mathscr{J}$-classes) and there is no $\mathscr{J}$-class K such that $I_4<K<J$. Then $I'\cup J$ is an ideal of S.

Case 1.2.1: $|J|\neq 4$. Take $I=I'\cup J=[J/I_5]$. Then by Lemma 3.12, $g_r(I)\leqslant\gamma(|I|-1)$.

Case 1.2.2: $|J|=4$. Since $|S|\neq 9$, $S\neq[J/I_5]$ so S has further $\mathscr{J}$-classes. Let K be a minimal $\mathscr{J}$-class of S among the $\mathscr{J}$-classes other than $\{0\}, I'\backslash\{0\}$ and J (under the usual ordering of $S/\mathscr{J}$). Then $I'\cup J\cup K$ is an ideal of S.

Case 1.2.2.1: $J \nleqslant K$.

If $|K| \neq 4$, we take $I = I_5 \cup K$. From Case 1.2.1, we have $g_r(I) \leqslant \gamma(|I| - 1)$. If $|K| = 4$, take $I = I_5 \cup J \cup K = [J : K/I_5]$. By Corollary 3.14, $g_r(I) \leqslant \gamma(|I| - 1)$.

Case 1.2.2.2: $J \leqslant K$. Take

$$I = I_5 \cup J \cup K = \begin{bmatrix} K \\ \overline{J} \\ \overline{I_5} \end{bmatrix}.$$

By Corollary 3.19, $g_r(I) \leqslant \gamma(|I| - 1)$.

Case 2: S has at least two 0-minimal ideals I_1 and I_2. If $|I_1| \neq 5$ or $|I_2| \neq 5$, let I be a 0-minimal ideal such that $|I| \neq 5$. Then from Case 1.1, we have $g_r(I) \leqslant \gamma(|I| - 1)$.

Suppose $|I_1| = |I_2| = 5$. Take $I = I_1 \cup I_2$. Then $|I| = 9$ and by Lemma 3.13, $g_r(I) \leqslant \max\{d(4), d(4)\} = 2 = \gamma(8) \leqslant \gamma(|I| - 1)$. $\square$

Theorem 3.21. *For each $n, g_r(n) = \gamma(n)$.*

Proof. (a) We first show that for each n, $g_r(n) \leqslant \gamma(n)$. If $|S| \leqslant 4$, then since S is regular, S is a union of subgroups, so $g_r(S) = 1$. Thus, $g_r(n) = 1 = \gamma(n)$, for $1 \leqslant n \leqslant 4$. Take any $n \geqslant 5$ and suppose the statement is true for $1, 2, \ldots, n-1$. Take any regular semigroup S of order n. We show that $g_r(S) \leqslant \gamma(n)$, from which we will have $g_r(n) \leqslant \gamma(n)$.

Case 1: $|S| \neq 5, |S| \neq 9$. By Lemma 3.20, S has an ideal I where $2 \leqslant |I|$ and $g_r(I) \leqslant \gamma(|I| - 1)$. If $S = I$, then since γ is non-decreasing, we have $g_r(S) = g_r(I) \leqslant \gamma(|I| - 1) \leqslant \gamma(|I|) = \gamma(n)$. So we suppose $S \neq I$. Since S/I is regular and $|S/I| < n$, the inductive hypothesis holds for S/I, that is, $g_r(S/I) \leqslant \gamma(|S/I|)$. By Result 2.6, we have

$$\begin{aligned} g_r(S) &\leqslant g_r(I) g_r(S/I) \\ &\leqslant \gamma(|I| - 1)\gamma(|S/I|) \\ &= \gamma(|I| - 1)\gamma(|S| - |I| + 1) \\ &\leqslant \gamma(|I| - 1 + |S| - |I| + 1) \quad \text{(by Lemma 3.5)} \\ &= \gamma(|S|). \end{aligned}$$

Case 2: $|S| = 5$. If S is completely 0-simple, it follows from Result 2.5 that $g_r(S) \leqslant d(4) = 2 = \gamma(5)$. Otherwise, let I be a 0-minimal ideal of S. Then $g_r(I) = g_r(S/I) = 1$ so $g_r(S) = 1 < 2 = \gamma(5)$.

Case 3: $|S| = 9$. If S is completely 0-simple, it follows from Result 2.5 $g_r(S) \leqslant d(8) = 2 < 3 = \gamma(9)$. Otherwise, let I be a 0-minimal ideal of S. Now for every 0-minimal ideal I of S such that $|I| \neq 5$, we have $|I| \leqslant 4$ or $|S/I| \leqslant 4$ so $g_r(I) = 1$ or $g_r(S/I) = 1$. From $g_r(8) = 2$, we have $g_r(I) g_r(S/I) \leqslant 2$, whence $g_r(S) < 3 = \gamma(9)$. If $|I| = 5$, it follows easily from Lemma 3.17 that $g_r(S) \leqslant 3 = \gamma(9)$.

(b) We now construct regular semigroups C_n such that $g_r(C_n) \geqslant \gamma(n)$.

Let C_1 be the one-element semigroup and for $2 \leqslant n \leqslant 4$, let $C_n = C_{n-1} \circ C_1$, the tower of C_{n-1} and C_1. Then $g_r(C_n) = 1 = \gamma(n)$ for $2 \leqslant n \leqslant 4$.

Let C_5 be a completely 0-simple semigroup with a non-group element a, for example the Brandt semigroup $\{a,b,c,d,0\}$ where $a^2=b^2=cd=dc=0$, $c^2=c=ab$, $d^2=d=ba$. Then a is a sequence in C_5 with no consecutive subsequence whose product is in a subgroup so $g_{\rm r}(C_5)\geqslant 2=\gamma(5)$.

For $6\leqslant n\leqslant 8$, let $C_n=C_5\circ C_{n-5}$. Then $g_{\rm r}(C_5)\geqslant 2=\gamma(n)$ for $6\leqslant n\leqslant 8$.

Let C_9 be the semigroup $[J_4/I_5]$ on nine elements given in the proof of Lemma 3.16, but where $J_4\varphi=\{g\}$ and $c^2\neq c$. Then a,c is a sequence in C_9 with no consecutive subsequence whose product is in a subgroup (since $ac=(a\varphi)c=gc=c$) so $g_{\rm r}(C_9)\geqslant 3=\gamma(9)$.

For $k\geqslant 2$ and $n=5k,5k+1,5k+2,5k+4$, let $C_n=C_5\circ C_{n-5}$. Let $C_{13}=C_{10}\circ C_3$. Then $g_{\rm r}(C_{13})=4=\gamma(13)$.

For $n=5k+3, k\geqslant 3$, let $C_n=C_9\circ C_{n-9}$.

We now show by induction that $g_{\rm r}(C_n)=\gamma(n)$. We assume that $n\geqslant 10$. Note that the only group elements of C_5, C_9 and C_{n-5} for $n\geqslant 10$ are its idempotent elements.

Case 1.a: $n=5k$; *then* $n-5=5(k-1)$.

Case 1.b: $n=5k+1$; *then* $n-5=5(k-1)+1$.

Case 1.c: $n=5k+2$; *then* $n-5=5(k-1)+2$. By Result 2.7 with $p=2$ and $q=2^{k-1}$, $C_n=C_5\circ C_{n-5}$ has a sequence of length $2\times 2^{k-1}-1=2^k-1$ with no consecutive subsequence whose product is in a subgroup. Hence $g_{\rm r}(C_n)\geqslant 2^k=\gamma(n)$.

Case 2: $n=5k+4$; then $n-5=5(k-1)+4$. By Result 2.7 with $p=2$ and $q=(\frac{3}{2})2^{k-1}$, $C_n=C_5\circ C_{n-5}$ has a sequence of length $q=2(\frac{3}{2})2^{k-1}-1=(\frac{3}{2})2^k-1$ with no consecutive subsequence whose product is in a subgroup. Hence $g_{\rm r}(C_n)\geqslant(\frac{3}{2})2^k=\gamma(n)$.

Case 3: $n=5k+3, k\geqslant 3$; then $n-9=5(k-2)+4$. By Result 2.7 with $p=3$ and $q=(\frac{3}{2})2^{k-2}$, $C_n=C_9\circ C_{n-9}$ has a sequence of length $3(\frac{3}{2})2^{k-2}-1=(\frac{9}{8})2^k-1$ with no consecutive subsequence whose product is in a subgroup. Hence $g_{\rm r}(C_n)\geqslant(\frac{9}{8})2^k=\gamma(n)$.

From parts (a) and (b) we have $g_{\rm r}(n)=\gamma(n)$. $\square$

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